Due Fri

## 1.3 – Matrices and Matrix Operations

Definition 1: A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The **size** of a matrix that has $m$ rows and $n$ columns is $m \times n$ (read "$m$ by $n$").

6. Use the following matrices to compute the indicated expression if it is defined.

$$\underset{3\times 2}{A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}}, \underset{2\times 2}{B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}}, \underset{2\times 3}{C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$tr(D) = 5$

a. $(2D^T - E)A$
b. $(4B)C + 2B$
c. $(-AC)^T + 5D^T$
d. $(BA^T - 2C)^T$
e. $B^T(CC^T - A^TA)$
f. $D^TE^T - (ED)^T$

Definition 7: If $A$ is any $m \times n$ matrix, then the **transpose of $A$**, denoted by $A^T$, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of $A$; that is, the first column of $A^T$ is the first row of $A$, the second column of $A^T$ is the second row of $A$, and so forth.

scalar multiplication

$$D^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} \qquad 2D^T = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix}$$

entry by entry

$$2D^T - E = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$(2D^T - E)A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$3 \times \boxed{3 \qquad 3} \times 2$$

these must match

size of the result

this is the row-column rule

$$= \begin{bmatrix} -4(3) - 3(-1) + 3(1) & -4(0) - 3(2) + 3(1) \\ 11(3) - 1(-1) + 2(1) & 11(0) - 1(2) + 2(1) \\ 0(3) + 1(-1) + 5(1) & 0(0) + 1(2) + 5(1) \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

$3 \times 2$

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Relabeling: $A = \begin{bmatrix} -4 & -3 & -3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$

<u>row vectors</u> of $A$ are

$$\vec{r}_1 = [-4 \quad -3 \quad -3]$$
$$\vec{r}_2 = [11 \quad -1 \quad 2]$$
$$\vec{r}_3 = [0 \quad 1 \quad 5]$$

Column vectors of $B$ are $\vec{c}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$AB$ can also be formed using the column method:

$$AB = \left[ A\vec{c}_1 \mid A\vec{c}_2 \right]$$

this is a partitioned matrix.

Note: $$A\vec{c}_1 = \underbrace{\underset{3\times 3}{\begin{bmatrix} -4 & -3 & -3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}} = \underset{3\times 1}{\begin{bmatrix} -6 \\ 36 \\ 4 \end{bmatrix}}$$

The row method is $$\begin{bmatrix} \vec{r}_1 \; B \\ \vec{r}_2 \; B \\ \vec{r}_3 \; B \end{bmatrix}$$

17. Use the column-row expansion of $AB$ to express this product as a sum of matrix products.

$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

The **column-row expansion** of $AB$ is $AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$, where $\mathbf{c}_i$ are column vectors of $A$ and $\mathbf{r}_i$ are row vectors of $B$.

$$AB = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & -9 & -3 \\ 2 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ 2 & -1 & 3 \end{bmatrix}$$

The linear system

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

Can be expressed using matrix multiplication.

When a linear system is written using a matrix product

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $A\mathbf{x} = \mathbf{b}$, $A$ is called the **coefficient matrix** of the system.

13. a. Express the matrix equation as a system of linear equations.

$$\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 5x_1 + 6x_2 - 7x_3 \\ -x_1 - 2x_2 + 3x_3 \\ 4x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{matrix} 5x_1 + 6x_2 - 7x_3 = 2 \\ -x_1 - 2x_2 + 3x_3 = 0 \\ 4x_2 - x_3 = 3 \end{matrix}$$

Definition 2: Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

Here, $A = \begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and

$\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

Note that $A\vec{x} = \begin{bmatrix} 5x_1 \\ -x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 6x_2 \\ -2x_2 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} -7x_3 \\ 3x_3 \\ -x_3 \end{bmatrix}$

$$= \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix} x_3$$

Definition 6: If $A_1, A_2, \dots, A_r$ are matrices of the same size, and if $c_1, c_2, \dots, c_r$ are scalars, then an expression of the form $c_1A_1 + c_2A_2 + \cdots + c_rA_r$ is called a **linear combination** of $A_1, A_2, \dots, A_r$ with **coefficients** $c_1, c_2, \dots, c_r$.

22. Example 6 of section 1.2 presents the linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

and the reduced row echelon form of its associated augmented matrix as

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$

$x_3 + 2x_4 = 0$

$x_6 = 0$

Let $r = x_2$, $s = x_4$, $t = x_5$

Express the solution as a linear combination of column vectors that contain only numerical entries.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t$$

**Theorem 1.3.1** If $A$ is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of $A$ in which the coefficients are the entries of $\mathbf{x}$.

The **main diagonal** of a square matrix contains entries from the upper left to lower right corners.

Definition 8: If $A$ is a square matrix, then the **trace of $\boldsymbol{A}$**, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of $A$. The trace of $A$ is undefined if $A$ is not a square matrix.